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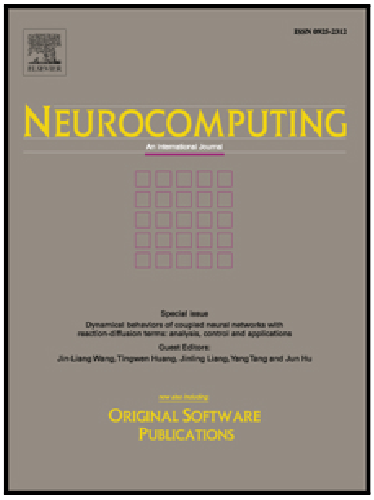

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Output-Feedback Tracking Control for Interval Type-2 Polynomial Fuzzy-Model-Based Control Systems

Bo Xiao[a,*], H.K. Lam[a], Ge Song[a], Hongyi Li[b]

[a]*Department of Informatics, King's College London, Strand, London, WC2R 2LS, United Kingdom*
[b]*College of Engineering, Bohai University, Jinzhou 121013, Liaoning, China*

Abstract

In this paper, the output tracking control issues of polynomial-fuzzy-model-based (PFMB) systems equipped with mismatched interval type-2 (IT2) membership functions are investigated. The output-feedback IT2 polynomial fuzzy controller connected with the nonlinear plant in a closed loop drives the system states of the nonlinear plant to track those of a stable reference model. The system stability is investigated based on the Lyapunov stability theory under the sum-of-squares (SOS)-based analysis approach and the SOS-based stability conditions are derived subjecting to an H_∞ performance. In addition, the fuzzy controller does not need to share the same membership functions with the plant. Moreover, the information of membership functions is included in the analysis to facilitate the analysis and relax the stability conditions. Numerical and experimental examples are presented to verify the effectiveness of the proposed tracking control approach.

Keywords: Fuzzy tracking control, Interval type-2 fuzzy logic, Polynomial fuzzy-model-based (PFMB) control systems, Stability analysis, Sum of squares (SOS).

*Corresponding author
Email addresses: bo.xiao@kcl.ac.uk (Bo Xiao), hak-keung.lam@kcl.ac.uk (H.K. Lam), ge.song@kcl.ac.uk (Ge Song), lihongyi2009@gamil.com (Hongyi Li)

1. Introduction

In the fuzzy tracking control design, a fuzzy controller is employed to drive the system states of the nonlinear plant to follow a reference or the system states of a stable reference model. Fuzzy tracking control problems are generally considered as more challenging than stabilization problems[45]. Takagi-Sugeno (T-S) fuzzy model, which plays an important role in the fuzzy-model-based (FMB) control system for its capability to provide general modeling frameworks for nonlinear systems, has been successfully adopted in the fuzzy tracking control design and related stability conditions have been obtained[45].

The fuzzy tracking control systems should firstly be guaranteed to be stable, where the Lyapunov stability theory is one of the most popular tools to investigate the system stability. In the analysis of the system stability based on the Lyapunov stability theory, the stability conditions of T-S FMB control systems can be formulated in the form of linear matrix inequalities (LMIs), which can be solved effectively through some numerical algorithms, for example, the interior point method[40]. If a common solution to all Lyapunov inequalities in terms of LMIs can be found, the control system is guaranteed to be asymptotically stable [46]. Considering the state feedback control, the most popular design method is the parallel distributed compensation (PDC) [46]. The basic idea of the PDC approach is that it requires the plant and controller both share the same premise fuzzy rule set. Through the PDC design approach, relaxed stability conditions can be achieved. In [41], the authors took the advantage of the symmetric property of the membership functions in the plant and the controller to relax the stability conditions. Inspired by the success of the work in [41], there are a lot of works managed to further relax the stability conditions [11, 28, 27, 44, 40, 6] and the asymptotically necessary and sufficient conditions for system stability are achieved by applying Pólya's Theorem [35].

Although, there are lots of successful applications of PDC design approach, the design flexibility is still restrained by the requirement that the plant and controller both share the same set of fuzzy rules. It makes sense to consider different fuzzy rules for the plant and controller, then it turns into a mismatched membership functions case, which means membership functions used in the fuzzy controller are different with those in the plant. It can render the system more flexibly and possibly reduce the control implementation cost by adopting mismatched membership premise function design approaches [17]. However, when the mismatched premise membership function approaches are adopted, the properties of matched membership functions in the PDC design approach vanish. In most of the related works, the information of membership functions has not been considered during the analysis, which means the stability conditions are valid unnecessarily for arbitrary membership functions. If the shapes of membership functions can be included in the analysis, the stability conditions can be only valid under the specific membership functions used in the applications and the stability conditions can be relaxed. There are some related work used the information of membership functions and obtained relaxed stability conditions can be found in[12, 13, 17].

Recently, the polynomial fuzzy model has been developed as an extension of the T-S fuzzy model[43]. In the polynomial fuzzy models, the polynomial terms are adopted in the modeling and analysis. When the order of the polynomial terms is zero, the polynomial fuzzy model is reduced to the T-S fuzzy model, so the T-S fuzzy model can be considered as a special case of the polynomial fuzzy model. Therefore, polynomial fuzzy model has more potential to a wider class of nonlinear systems. However, due to the polynomial terms, the LMI approaches used for the T-S fuzzy model cannot be used for the polynomial fuzzy model to obtain the stability conditions. Instead, the sum of squares (SOS) approach is adopted in the stability analysis and the stability conditions can be developed in terms of SOS, which can be further solved efficiently by a third party Matlab toolbox SOSTOOLS, more details regarding the toolbox can be found in the manual of SOSTOOLS[34]. Given that the concept of polynomial fuzzy model does not have a long history, there are relatively less work on it when comparing with the T-S fuzzy counterpart. In [43], SOS techniques were first adopted in the polynomial-fuzzy-model-based (PFMB) control system to achieve stability conditions, and other related works can be found in [43, 42, 36, 33, 13, 14, 17, 19, 20, 5, 22, 23].

The most widely used membership functions are of type-1 fuzzy set. The type-1 fuzzy set has been successfully applied to tackle the nonlinearities in control systems but it lacks the ability to handle the uncertainties directly[32, 31, 16]. Under many situations, the uncertainties are inevitable and can be easily found during the construction of the rules in FMB control systems. Generally, the uncertainties can be classified into two types, the linguistic uncertainties and random uncertainties[31]. Because of different comprehension from different people, there are linguistic uncertainties in defining the membership functions, and some unavoidable mistakes and the limitation of instrument precision also result in random uncertainties. In order to include the uncertainties into the membership functions, the concept of footprint of uncertainty (FOU) has been introduced to the type-1 membership functions and then the type-1 membership functions are transformed into type-2 membership functions [31].

The general form of FOU in type-2 membership functions is a function of premise variables and can be regarded as a set of type-1 membership functions. However, there are huge complexities lying in the FOU, which further make the stability analysis very complex and the numerical simulations will also be time-consuming. For this reason, the most popular used type-2 membership functions are interval type-2 (IT2) membership functions. In terms of IT2 membership function, the secondary grades of type-2 membership functions are constants instead of functions of premise variables. Through IT2 membership functions, we can not only handle the uncertainties but can also reduce the computational burden [10, 26, 31, 3]. Recently, the research on the system control and stability analysis has been conducted based on the framework of IT2 fuzzy systems, which can be found in [16, 9, 2, 7, 8, 18, 24, 25].

In the work in [4, 29], the robust adaptive tracking control problem is handled through fuzzy approaches; in the work in [45, 49, 1], the output feedback tracking control issue is investigated through T-S FMB approach; in the work in [15], the investigation on the PFMB tracking control problem is based on type-1 fuzzy logic. Having mentioned the previous works, although there are some work on the tracking control issues based on fuzzy approaches, to the best knowledge of the author, the stability and performance of the IT2 PMFB tracking control systems have not been colorgreenfully investigated.

As mentioned above, although there are works on the fuzzy tracking control problems, the tracking control issues based on IT2 PFMB control systems with mismatched membership functions are rarely investigated. In this paper, the tracking control problems of IT2 PFMB control systems is considered under the SOS-based framework and the stability conditions are obtained in terms of SOS. Based on IT2 fuzzy logic, the uncertainties of the control systems can be handled directly, also unlike the full-state feedback control approach [45, 49], the tracking control approach proposed in this paper is implemented through the system output only, which makes the control systems more flexible and more convenient to be applied to real applications. In addition, the information of membership functions is considered in the analysis to further relax the stability conditions.

Given that the IT2 membership functions are generally continuous, which will result in the infinite number of stability conditions and then it is not practical to be solved numerically. To tackle this difficulty, the whole operation domain of membership functions is firstly divided into some sub-domains, then we construct the lower and upper bounds of the FOU within every sub-domain. All of these lower and upper bounds are in forms of linear functions, which can be easily included in the SOS-based conditions and then further be solved through SOSTOOLS. The H_∞ performance is one of the most popular ways to improve the control performance, the applications of H_∞ performance on fuzzy control systems can be viewed in [45, 21, 38, 15, 37, 47, 48], also the Hankel-norm performance in this paper, the SOS-based stability conditions are derived to guarantee the system stability subject to an H_∞ performance.

The rest of the paper is organized as follows. Section 2 briefly presents the preliminaries of IT2 polynomial fuzzy models and controllers, also the connection with the reference model. In Section 3, the stability issues of IT2 PFMB tracking control system are discussed based on the Lyapunov stability theory and also the analysis of H_∞ performance. In Section 4, simulation examples are given to illustrate the merits of proposed tracking control method. Conclusions are drawn in Section 5.

2. Interval type-2 Polynomial Fuzzy Model, Reference Model and Fuzzy Controller

2.1. IT2 Polynomial Fuzzy Model

An IT2 polynomial fuzzy model with p rules is employed to describe the dynamics of the nonlinear plant[43, 15]. The rules are of the following format where the antecedents are IT2 fuzzy sets and the consequent is a polynomial system:

$$\text{Rule } i: \text{IF } f_1(\mathbf{y}(t)) \text{ is } \tilde{M}_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\mathbf{y}(t)) \text{ is } \tilde{M}_\Psi^i$$

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t), \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\hat{\mathbf{x}}(\mathbf{x}(t)) \tag{2}$$

where $\tilde{M}_\alpha^i$ is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{y}(t))$, $\alpha = 1, 2, \ldots, \Psi$ and $i = 1, 2, \ldots, p$; Ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n\times m}$ are known polynomial system and input matrices; $\mathbf{x}(t) \in \Re^n$ is the system-state vector, $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}(t)$, and $\mathbf{u}(t) \in \Re^m$ is the control input vector, $\mathbf{C} \in \Re^{q\times N}$ is the constant output matrix, $\mathbf{y}(t) \in \Re^q$ is the system output vector. The membership grade function of the i-th rule is within the following interval sets:

$$\tilde{w}_i(\mathbf{y}(t)) \in [\prod_{\alpha=1}^{\Psi} \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t))), \prod_{\alpha=1}^{\Psi} \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t)))], \quad i = 1, 2, \ldots, p \tag{3}$$

and we define

$$w_i^L(\mathbf{y}(t)) = \prod_{\alpha=1}^{\Psi} \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t))), \tag{4}$$

$$w_i^U(\mathbf{y}(t)) = \prod_{\alpha=1}^{\Psi} \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t))) \tag{5}$$

in which $0 \leq \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t))) \leq 1$ and $0 \leq \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t))) \leq 1$ denote the upper and lower grades of membership governed by their upper and lower membership functions, respectively. By the definition of IT2 membership functions, the property $0 \leq \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t))) \leq \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{y}(t))) \leq 1$ holds, which further leads to $0 \leq w_i^L(\mathbf{y}(t)) \leq w_i^U(\mathbf{y}(t)) \leq 1$ for all i.

Also we define $\tilde{w}_i(\mathbf{y}(t))$ as follows:

$$\tilde{w}_i(\mathbf{y}(t)) = \underline{\lambda}_i(\mathbf{y}(t))w_i^L(\mathbf{y}(t)) + \overline{\lambda}_i(\mathbf{y}(t))w_i^U(\mathbf{y}(t)), \tag{6}$$

$$0 \leq \underline{\lambda}_i(\mathbf{y}(t)) \leq 1, \tag{7}$$

$$0 \leq \overline{\lambda}_i(\mathbf{y}(t)) \leq 1, \tag{8}$$

$$\underline{\lambda}_i(\mathbf{y}(t)) + \overline{\lambda}_i(\mathbf{y}(t)) = 1, \forall\, i \tag{9}$$

where $\underline{\lambda}_i(\mathbf{y}(t))$ and $\overline{\lambda}_i(\mathbf{y}(t))$ are nonlinear functions to be determined.

The IT2 polynomial fuzzy model is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \tilde{w}_i(\mathbf{y}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)) \tag{10}$$

where

$$\sum_{i=1}^{p} \tilde{w}_i(\mathbf{y}(t)) = 1, \quad \tilde{w}_i(\mathbf{y}(t)) \geq 0\ \forall\ i. \tag{11}$$

2.2. Reference Model

A stable reference model is defined as follows:

$$\begin{aligned} \dot{\mathbf{x}}_\mathbf{r}(t) &= \mathbf{A}_\mathbf{r}\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t)) + \mathbf{B}_\mathbf{r}\mathbf{r}(t), \\ \mathbf{y}_\mathbf{r}(t) &= \mathbf{C}\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t)) \end{aligned} \tag{12}$$

where $\mathbf{x}_\mathbf{r}(t) \in \Re^n$ is the state vector of the reference model, which needs to be followed by the fuzzy model, $\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}_\mathbf{r}(t)$ as the entries, $\mathbf{A}_\mathbf{r} \in \Re^{n\times N}$ and $\mathbf{B}_\mathbf{r} \in \Re^{n\times m}$ are the constant system and input matrices, respectively, $\mathbf{r}(t) \in \Re^m$ is the reference input vector, $\mathbf{y}_\mathbf{r}(t) \in \Re^q$ is the output vector of the reference model.

2.3. IT2 Output-Feedback Polynomial Fuzzy Controller

An output-feedback polynomial fuzzy controller is proposed to drive the system states of the nonlinear plant in the form of (10) to follow those of the stable reference model (12).

Define the state error in polynomial form as

$$\hat{\mathbf{e}}(t) = \hat{\mathbf{x}}(\mathbf{x}(t)) - \hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t)). \tag{13}$$

From (10), (12) and (13), the output error is defined as follows:

$$\mathbf{e}_{\mathbf{y}}(t) = \mathbf{y}(t) - \mathbf{y}_{\mathbf{r}}(t) = \mathbf{C}\hat{\mathbf{e}}(t). \tag{14}$$

An output-feedback IT2 polynomial fuzzy controller with c rules is employed to stabilise the plant represented by the IT2 polynomial fuzzy model (10). The format of the IT2 polynomial fuzzy controller is as follows:

$$\begin{aligned}\text{Rule } j : \text{IF } g_1(\mathbf{y}(t)) \text{ is } \tilde{N}_1^j \text{ AND } \cdots \text{ AND } g_{\Omega}(\mathbf{y}(t)) \text{ is } \tilde{N}_{\Omega}^j \\ \text{THEN } \mathbf{u}(t) = \mathbf{F}_j(\mathbf{h}(t))\mathbf{e}_{\mathbf{y}}(t) + \mathbf{G}_j(\mathbf{h}(t))\mathbf{y}_{\mathbf{r}}(t)\end{aligned} \tag{15}$$

where $\tilde{N}_{\beta}^j$ is an IT2 fuzzy term of rule j corresponding to function $g_{\beta}(\mathbf{y}(t))$, where $\beta = 1, 2, \ldots, \Omega$ and $j = 1, 2, \ldots, c$; Ω is a positive integer. Define $\mathbf{h}(t) = [\mathbf{y}(t) \quad \mathbf{y}_{\mathbf{r}}(t)]$. $\mathbf{F}_j(\mathbf{h}(t)) \in \Re^{m\times q}$ and $\mathbf{G}_j(\mathbf{h}(t)) \in \Re^{m\times q}$, $j = 1, 2, \ldots, c$, are the polynomial gains to be determined. Along the same way in fuzzy model, the membership grade function of the j-th rule is within the following interval sets:

$$\tilde{m}_j(\mathbf{y}(t)) \in [\prod_{\beta=1}^{\Omega} \underline{\mu}_{\tilde{N}_{\beta}^j}(g_{\beta}(\mathbf{y}(t))), \prod_{\beta=1}^{\Omega} \overline{\mu}_{\tilde{N}_{\beta}^j}(g_{\beta}(\mathbf{y}(t)))], \quad j = 1, 2, \ldots, c \tag{16}$$

and we define

$$m_j^L(\mathbf{y}(t)) = \prod_{r=1}^{\Omega} \underline{\mu}_{\tilde{N}_{\beta}^j}(g_{\beta}(\mathbf{y}(t))), \tag{17}$$

$$m_j^U(\mathbf{y}(t)) = \prod_{r=1}^{\Omega} \overline{\mu}_{\tilde{N}_{\beta}^j}(g_{\beta}(\mathbf{y}(t))) \tag{18}$$

in which $0 \leq \overline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{y}(t))) \leq 1$ and $0 \leq \underline{\mu}_{\tilde{N}_\beta^i}(g_\beta(\mathbf{y}(t))) \leq 1$ denote the upper and lower grades of membership governed by the upper and lower membership functions, respectively. By the definition of IT2 membership functions, the property $0 \leq \underline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{y}(t))) \leq \overline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{y}(t))) \leq 1$ holds and further leads to the $0 \leq m_j^L(\mathbf{y}(t)) \leq m_j^U(\mathbf{y}(t)) \leq 1$ valid for all j.

Also we define $\tilde{m}_j(\mathbf{y}(t))$ as follows:

$$\tilde{m}_j(\mathbf{y}(t)) = \frac{\underline{\kappa}_j(\mathbf{y}(t))m_j^L(\mathbf{y}(t)) + \overline{\kappa}_j(\mathbf{y}(t))m_j^U(\mathbf{y}(t))}{\sum_{k=1}^{c}\Big(\underline{\kappa}_k(\mathbf{y}(t))m_k^L(\mathbf{y}(t)) + \overline{\kappa}_k(\mathbf{y}(t))m_k^U(\mathbf{y}(t))\Big)} \geq 0, \tag{19}$$

$$0 \leq \underline{\kappa}_j(\mathbf{y}(t)) \leq 1, \tag{20}$$

$$0 \leq \overline{\kappa}_j(\mathbf{y}(t)) \leq 1, \tag{21}$$

$$\underline{\kappa}_j(\mathbf{y}(t)) + \overline{\kappa}_j(\mathbf{y}(t)) = 1 \;\forall\; j \tag{22}$$

where $\underline{\kappa}_j(\mathbf{y}(t))$ and $\overline{\kappa}_j(\mathbf{y}(t))$ are nonlinear functions to be determined.

The IT2 polynomial fuzzy controller is described by

$$\mathbf{u}(t) = \sum_{j=1}^{c} \tilde{m}_j(\mathbf{y}(t))(\mathbf{F}_j(\mathbf{h}(t))\mathbf{C}\hat{\mathbf{e}}(t) + \mathbf{G}_j(\mathbf{h}(t))\mathbf{y}_\mathbf{r}(t)) \tag{23}$$

where

$$\sum_{i=1}^{c} \tilde{m}_j(\mathbf{y}(t)) = 1, \quad \tilde{m}_j(\mathbf{y}(t)) \geq 0 \;\forall\; j. \tag{24}$$

3. Stability Analysis

For brevity, in the following analysis, $\tilde{w}_i(\mathbf{y}(t))$ is denoted as $\tilde{w}_i$ and $\tilde{m}_j(\mathbf{y}(t))$ is denoted as $\tilde{m}_j$, also the time t associated with the variables is dropped for the situation without ambiguity, e.g., $\mathbf{h}(t)$, $\mathbf{x}(t)$, $\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t))$ and $\hat{\mathbf{x}}(\mathbf{x}(t))$ are denoted as $\mathbf{h}$, $\mathbf{x}$, $\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r})$ and $\hat{\mathbf{x}}(\mathbf{x})$, respectively.

Considering the polynomial fuzzy model (10) and the output-feedback polynomial fuzzy controller (23), we have the following close-loop dynamic equation:

$$\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{e}} + \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}), \tag{25}$$

in which $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and $\hat{\mathbf{x}}(\mathbf{x}) = [\hat{x}_1(\mathbf{x}), \hat{x}_2(\mathbf{x}), \ldots, \hat{x}_N(\mathbf{x})]$.

The relationship between $\dot{\hat{\mathbf{x}}}$ and $\dot{\mathbf{x}}$ is as follows:

$$\dot{\hat{\mathbf{x}}} = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x})\dot{\mathbf{x}}, \tag{26}$$

in which $\mathbf{T}(\mathbf{x}) \in \Re^{N\times n}$ with its $\alpha\beta$-th element $T_{\alpha\beta}(\mathbf{x})$ defined as

$$T_{\alpha\beta}(\mathbf{x}) = \frac{\partial \hat{x}_\alpha(\mathbf{x})}{\partial x_\beta}, \alpha = 1, 2, \ldots, N; \beta = 1, 2, \ldots, n. \tag{27}$$

Through combining (25) with (26), the polynomial dynamic model can be obtained as follows:

$$\dot{\hat{\mathbf{x}}} = \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{e}} + \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}). \tag{28}$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$, $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$. Similarly, denote $\mathbf{x}_{\mathbf{r}} = [x_{r_1}, x_{r_2}, \ldots, x_{r_n}]^T$ and $\hat{\mathbf{x}}(\mathbf{x}_{\mathbf{r}}) = [\hat{x}_{r_1}(\mathbf{x}_{\mathbf{r}}), \hat{x}_{r_2}(\mathbf{x}_{\mathbf{r}}), \ldots, \hat{x}_{r_N}(\mathbf{x}_{\mathbf{r}}),]^T$. From (12), we have the polynomial dynamic model for the reference model:

$$\dot{\hat{\mathbf{x}}}(\mathbf{x}_{\mathbf{r}}) = \frac{\partial \hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})}{\partial \mathbf{x}_{\mathbf{r}}} \frac{d\mathbf{x}_{\mathbf{r}}}{dt} = \mathbf{H}(\mathbf{x}_{\mathbf{r}})\dot{\mathbf{x}}_{\mathbf{r}} = \tilde{\mathbf{A}}_{\mathbf{r}}\hat{\mathbf{x}}(\mathbf{x}_{\mathbf{r}}) + \tilde{\mathbf{B}}_{\mathbf{r}}\mathbf{r} \tag{29}$$

where $\tilde{\mathbf{A}}_{\mathbf{r}} = \mathbf{H}(\mathbf{x}_{\mathbf{r}})\mathbf{A}_{\mathbf{r}}$, $\tilde{\mathbf{B}}_{\mathbf{r}} = \mathbf{H}(\mathbf{x}_{\mathbf{r}})\mathbf{B}_{\mathbf{r}}$ and $\mathbf{H}(\mathbf{x}_{\mathbf{r}}) \in \Re^{N\times n}$ with its $\alpha\beta$-th element is defined as

$$H_{\alpha\beta}(\mathbf{x}_{\mathbf{r}}) = \frac{\partial \hat{x}_{r_\alpha}(\mathbf{x})}{\partial x_{r_\beta}}, \alpha = 1, 2, \ldots, N; \beta = 1, 2, \ldots, n. \tag{30}$$

From the polynomial dynamic models for the plant and reference, the state error can be achieved as

$$\begin{aligned}\dot{\hat{\mathbf{e}}} &= \dot{\hat{\mathbf{x}}}(\mathbf{x}) - \dot{\hat{\mathbf{x}}}(\mathbf{x}_{\mathbf{r}}) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{e}} \\ &+ \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}} + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) - \tilde{\mathbf{B}}_{\mathbf{r}}\mathbf{r}.\end{aligned} \tag{31}$$

3.1. Basic Stability Analysis

To facilitate the stability analysis of error system (31), we define an augmented vector $\hat{\mathbf{v}} = \mathbf{\Gamma}^{-1}\hat{\mathbf{e}}$, where $\mathbf{\Gamma} = [\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} \quad ortc(\mathbf{C}^T)] \in \Re^{N\times N}$ and

$ortc(\mathbf{C}^T)$ denotes the orthogonal complement of $\mathbf{C}^T$[15, 30]. Consequently, we have $\mathbf{C}\boldsymbol{\Gamma} = [\mathbf{I}_l \quad \mathbf{0}]$, where $\mathbf{I}_q \in \Re^{q\times q}$ is the identity matrix.

Furthermore, we define $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T = \begin{bmatrix} \mathbf{X}_{11} & 0 \\ 0 & \mathbf{X}_{22}(\tilde{\mathbf{x}}) \end{bmatrix} \in \Re^{N\times N}$ [30, 15], $\mathbf{X}_{11} \in \Re^{q\times q}$ and $\mathbf{X}_{22}(\tilde{\mathbf{x}}) \in \Re^{(N-q)\times(N-q)}$; $\tilde{\mathbf{x}} = (x_{j_1}, x_{j_2}, \dots, x_{j_q}, x_{r_{k_1}}, x_{r_{k_2}}, \dots, x_{r_{k_s}})$; the row indices $\mathbf{J} = \{j_1, j_2, \dots, j_q\}$ and $\mathbf{K} = \{k_1, k_2, \dots, k_s\}$ are the rows indicating that the entire row of $\mathbf{B}_i(\mathbf{x_r})$ and $\mathbf{B_r}(\mathbf{x})$ are all zeros, respectively [43]. As $\mathbf{X}(\tilde{\mathbf{x}})$ is required to be positive definite, it implies that the inverse of $\mathbf{X}_{11}$ and $\mathbf{X}_{22}(\tilde{\mathbf{x}})$ exist.

Using the fact that $\mathbf{F}_j(\mathbf{h})\mathbf{C}\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) = [\mathbf{M}_j(\mathbf{h}) \quad \mathbf{0}]$ and $\mathbf{G}_j(\mathbf{h})\mathbf{C}\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) = [\mathbf{N}_j(\mathbf{h}) \quad \mathbf{0}]$, where $\mathbf{M}_j(\mathbf{h}) = \mathbf{F}_j(\mathbf{h})\mathbf{X}_{11} \in \Re^{m\times q}$ and $\mathbf{N}_j(\mathbf{h}) = \mathbf{G}_j(\mathbf{h})\mathbf{X}_{11} \in \Re^{m\times q}$, it follows from (31) and the augmented vector $\hat{\mathbf{v}}$ that we obtain the augmented system dynamics $\dot{\hat{\mathbf{v}}}$ as follows.

$$
\begin{aligned}
\dot{\hat{\mathbf{v}}} &= \boldsymbol{\Gamma}^{-1}\dot{\hat{\mathbf{e}}} \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\boldsymbol{\Gamma}^{-1}\tilde{\mathbf{A}}_i(\mathbf{x}) + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{e}} \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\boldsymbol{\Gamma}^{-1}(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_\mathbf{r}) + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C})\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r}) - \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_\mathbf{r}\mathbf{r} \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\boldsymbol{\Gamma}^{-1}\tilde{\mathbf{A}}_i(\mathbf{x})\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C}\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}))\mathbf{X}(\tilde{\mathbf{x}})^{-1}\boldsymbol{\Gamma}^{-1}\hat{\mathbf{e}} \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\boldsymbol{\Gamma}^{-1}(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_\mathbf{r})\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C}\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}))\mathbf{X}(\tilde{\mathbf{x}})^{-1}\boldsymbol{\Gamma}^{-1}\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r}) - \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_\mathbf{r}\mathbf{r} \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\boldsymbol{\Gamma}^{-1}\tilde{\mathbf{A}}_i(\mathbf{x})\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x}) \times [\mathbf{M}_j(\mathbf{h}) \quad \mathbf{0}])\mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}} \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j(\boldsymbol{\Gamma}^{-1}(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_\mathbf{r})\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x}) \times [\mathbf{N}_j(\mathbf{h}) \quad \mathbf{0}])\mathbf{X}(\tilde{\mathbf{x}})^{-1}\boldsymbol{\Gamma}^{-1}\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r}) - \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_\mathbf{r}\mathbf{r} \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i\tilde{m}_j\boldsymbol{\Phi}_{ij}(\mathbf{x}, \mathbf{x_r})\mathbf{z}, \qquad (32)
\end{aligned}
$$

where $\boldsymbol{\Phi}_{ij}(\mathbf{x}, \mathbf{x_r}) = [\boldsymbol{\Phi}_{ij}^{(1)}(\mathbf{x}, \mathbf{x_r}) \quad \boldsymbol{\Phi}_{ij}^{(2)}(\mathbf{x}, \mathbf{x_r}) \quad \boldsymbol{\Phi}_{ij}^{(3)}(\mathbf{x}, \mathbf{x_r})]$, $\boldsymbol{\Phi}_{ij}^{(1)}(\mathbf{x}, \mathbf{x_r}) = \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{A}}_i(\mathbf{x})\boldsymbol{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x}) \times [\mathbf{M}_j(\mathbf{h}) \quad \mathbf{0}]$, $\boldsymbol{\Phi}_{ij}^{(2)}(\mathbf{x}, \mathbf{x_r}) = \boldsymbol{\Gamma}^{-1}(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_\mathbf{r})\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x}) \times [\mathbf{N}_j(\mathbf{h}) \quad \mathbf{0}]$,

$$\boldsymbol{\Phi}_{ij}^{(3)}(\mathbf{x},\mathbf{x_r}) = -\boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}_{\mathbf{r}},\ \mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}} \\ \mathbf{X}(\tilde{\mathbf{x}})^{-1}\boldsymbol{\Gamma}^{-1}\mathbf{x_r} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{X}(\tilde{\mathbf{x}})^{-1}\boldsymbol{\Gamma}^{-1}\hat{\mathbf{e}} \\ \mathbf{X}(\tilde{\mathbf{x}})^{-1}\boldsymbol{\Gamma}^{-1}\mathbf{x_r} \\ \mathbf{r} \end{bmatrix}.$$

Before proceeding further, the following lemma is introduced to support the stability analysis.

Lemma 1. *For any invertible polynomial matrix $\mathbf{X}(\tilde{\mathbf{x}})$, the following lemma holds:*

$$\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} = -\mathbf{X}(\tilde{\mathbf{x}})^{-1}\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{X}(\tilde{\mathbf{x}})^{-1}. \tag{33}$$

Proof. Given that

$$\frac{\partial \mathbf{I}}{dx_k} = 0, \tag{34}$$

replacing $\mathbf{I}$ by $\mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{X}(\tilde{\mathbf{x}})$, (34) can be rewritten as

$$\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} = 0. \tag{35}$$

It follows that

$$\mathbf{X}(\tilde{\mathbf{x}})^{-1}\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} + \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k}\mathbf{X}(\tilde{\mathbf{x}}) = 0. \tag{36}$$

Rearranging terms will lead to (33). □

With Lemma 1, the term $\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt}$ appearing in the following analysis can be written as follows.

$$\begin{aligned} \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} &= \sum_{k=1}^{n}(\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k}\dot{x}_k + \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_{rk}}\dot{x}_{rk}) \\ &= -\sum_{k\in\mathbf{J}}\mathbf{X}(\tilde{\mathbf{x}})^{-1}\Big(\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\sum_{i=1}^{p} w_i \mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})\Big)\mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ &\quad - \sum_{k\in\mathbf{K}}\mathbf{X}(\tilde{\mathbf{x}})^{-1}\Big(\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_{rk}}\mathbf{A}_r^{(k)}(\mathbf{x})\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x_r})\Big)\mathbf{X}(\tilde{\mathbf{x}})^{-1}. \end{aligned} \tag{37}$$

Consider the following polynomial Lyapunov function candidate to investigate the stability of the augmented system (32).

$$V(t) = \hat{\mathbf{v}}^T\mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}}. \tag{38}$$

It follows from (32) and (38) that we have

$$\begin{aligned}\dot{V}(t) &= \dot{\hat{\mathbf{v}}}^T\mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}} + \hat{\mathbf{v}}^T\mathbf{X}(\tilde{\mathbf{x}})^{-1}\dot{\hat{\mathbf{v}}} + \hat{\mathbf{v}}^T\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt}\hat{\mathbf{v}} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{w}_i\tilde{m}_j\mathbf{z}^T\mathbf{\Phi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z}_1 + \mathbf{z}_1^T\sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{w}_i\tilde{m}_j\mathbf{\Phi}_{ij}\mathbf{z} \qquad (39)\\ &+ \mathbf{z}_1^T\Big(\sum_{k\in\mathbf{J}}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) - \sum_{k\in\mathbf{K}}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_{rk}}\mathbf{A}_r^{(k)}(\mathbf{x})\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r})\Big)\mathbf{z}_1 \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{w}_i\tilde{m}_j\mathbf{z}^T\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z} - \mathbf{z}_1^T\mathbf{z}_1 + \sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3 \qquad (40)\end{aligned}$$

where $\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) = \begin{bmatrix}\mathbf{\Xi}_{ij}^{(11)}(\mathbf{x},\mathbf{x_r}) & * & * \\ \mathbf{\Phi}_{ij}^{(2)}(\mathbf{x},\mathbf{x_r})^T & -\sigma_1^2\mathbf{I} & * \\ \mathbf{\Phi}_{ij}^{(3)}(\mathbf{x},\mathbf{x_r})^T & \mathbf{0} & -\sigma_2^2\mathbf{I}\end{bmatrix}$, $\mathbf{\Xi}_{ij}^{11}(\mathbf{x},\mathbf{x_r}) = \mathbf{\Phi}_{ij}^{(1)}(\mathbf{x},\mathbf{x_r}) + \mathbf{\Phi}_{ij}^{(1)}(\mathbf{x},\mathbf{x_r})^T + \mathbf{I} - \sum_{k\in\mathbf{J}}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) - \sum_{k\in\mathbf{K}}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_{rk}}\mathbf{A}_r^{(k)}(\mathbf{x})\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r})$.

When

$$\sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{w}_i\tilde{m}_j\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) < 0, \qquad (41)$$

we have

$$\dot{V}(t) \le -\mathbf{z}_1^T\mathbf{z}_1 + \sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3. \qquad (42)$$

Considering the termination time of control t_f and taking integration on both sides of (42) with respect to time t, we obtain the following H_∞ performance:

$$\frac{\int_0^{t_f}\mathbf{z}_1^T\mathbf{z}_1 - V(0)}{\int_0^{t_f}(\sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3)dt} \le 1 \qquad (43)$$

where the tracking performance can be improved with smaller values of $\sigma_1 > 0$ and $\sigma_2 > 0$. In the definition of H_∞ in (43), $\mathbf{z}_1$ represents the tracking error, which should be attenuated against the system input, which represented by $\mathbf{z}_2$, $\mathbf{z}_3$. It can be found that from the definition in (43), when σ_1 and σ_2 are reduced, smaller $\int_0^{tf}\mathbf{z}_1^T\mathbf{z}_1$ is required to keep (43) valid, then the tracking error represented by $\mathbf{z}_1$ is attenuated.

In order to ensure (41) to be valid, the basic stability condition can be derived by requiring $\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) < 0$ for all i and j. The basic results can be summarized as the following theorem[15]:

Theorem 1. *The IT2 PFMB system (25), which is formed by a nonlinear plant represented by the IT2 polynomial fuzzy model and the IT2 polynomial fuzzy controller connected in a closed loop, in which the states are driven to follow those of the stable reference model (12) subject to H_∞ performance (43) if there exist polynomial matrices $\mathbf{M}_j(\mathbf{h}) \in \Re^{m\times q}$, $\mathbf{N}_j(\mathbf{h}) \in \Re^{m\times q}$, $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$ such that the following SOS-based conditions are satisfied:*

$$\begin{aligned} &\nu^T(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I})\nu \text{ is SOS;} \\ &-\rho^T(\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) + \varepsilon_2(\mathbf{x},\mathbf{x_r})\mathbf{I})\rho \text{ is SOS} \quad \forall i,j \end{aligned} \tag{44}$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$ and $\mathbf{x_r}$, $\rho \in \Re^{2N+m}$ is an arbitrary vector independent of $\mathbf{x}$ and $\mathbf{x_r}$, $\varepsilon_1(\tilde{\mathbf{x}}) > 0$, $\varepsilon_2(\mathbf{x},\mathbf{x_r}) > 0$ are predefined scalar polynomials.

Remark 1. *Referring to Theorem 1, the number of SOS variables is $2c+1$ and the number of SOS based stability conditions is $2pc+1$. From Theorem 1, it can be found that the stability conditions are clear and straightforward. However, the information of the membership functions has not been included in the conditions, which means the stability conditions in Theorem 1 are unnecessarily valid for all kinds of membership functions. In the real application, only the specific membership functions adopted in the plants and controllers need to be considered. Therefore, there is conservativeness lying in the basic stability conditions. In order to reduce the conservativeness and further relax the stability conditions, the membership-function-dependent analysis will be introduced in the following section.*

3.2. Membership-Function-Dependent Stability Analysis

In order to guarantee the stability of the system, the stability condition (41) has to be satisfied. However, as $\tilde{w}_i\tilde{m}_j \equiv \tilde{h}_{ij}(\mathbf{y})$ is a function of $\mathbf{y}$, the stability condition (41) has to be satisfied for all values of membership grades implying an infinite number of stability conditions. Consequently, when the membership functions $\tilde{h}_{ij}(\mathbf{y})$ are incorporated into the stability conditions, it is not practical to find a feasible solution to the stability conditions of infinite number. In this paper, we propose a technique to bring the information of membership functions into the stability analysis, which avoids turning the number of stability conditions into infinity but still can achieve more relaxed stability conditions.

To facilitate the stability analysis and bring the information of membership functions into the analysis, a discretization process is applied to the membership functions. The whole operating domain Φ is divided into L connected sub-domains, Φ_l, $l = 1, 2, \ldots, L$ such that $\Phi = \bigcup_{l=1}^{L} \Phi_l$. We denote the portion of $\tilde{h}_{ij}(\mathbf{y})$ where $\mathbf{y} \in \Phi_l$ (the portion of $\tilde{h}_{ij}(\mathbf{y})$ in the l-th sub-domain) as $\tilde{h}_{ijl}$ such that $\tilde{h}_{ij}(\mathbf{y}) = \bigcup_{l=1}^{L} \tilde{h}_{ijl}(\mathbf{y})$. Then we can construct the linear and upper linear function in very sub-domain, which guarantee that the FOU in sub-domains is between the upper and lower linear functions.

In the following, we conduct the stability analysis sub-domain by sub-domain by utilizing the information of $\tilde{h}_{ijl}(\mathbf{y})$ for $\mathbf{y} \in \Phi_l$. Once the control system operated in every sub-domain is guaranteed to be stable, the whole control system is guaranteed to be a stable one. We can then rewrite the basic stability condition in the l-th sub-domain as follows:

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{h}_{ijl}(\mathbf{y})\mathbf{z}^T\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z}\\
&=\sum_{i=1}^{p}\sum_{j=1}^{c}(\hat{h}_{ijl}(\mathbf{y})+\tilde{h}_{ijl}(\mathbf{y})-\hat{h}_{ijl}(\mathbf{y}))\mathbf{z}^T\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z}<0, \mathbf{x}\in\Phi_l, l=1,\,2,\ldots,L
\end{aligned} \tag{45}$$

where $\hat{h}_{ijl}(\mathbf{y}) \geq 0$ is a function, which is an estimate of $h_{ijl}(\mathbf{y})$ to be determined. Meanwhile, we define some non-negative matrices $\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x_r}) = \mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x_r})^T \in \Re^{(2N+m)\times(2N+m)} \geq 0$, which is required to satisfy the condition that $\mathbf{Y}_{ijl}(\mathbf{x},\mathbf{x_r}) \geq \mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})$. From (45), we have

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{h}_{ijl}(\mathbf{y})\mathbf{z}^T\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z} &\leq \sum_{i=1}^{p}\sum_{j=1}^{c}\hat{h}_{ijl}(\mathbf{y})\mathbf{z}^T\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z}\\ &+\sum_{i=1}^{p}\sum_{j=1}^{c}|\tilde{h}_{ijl}(\mathbf{y})-\hat{h}_{ijl}(\mathbf{y})|\mathbf{z}^T\mathbf{Y}_{ijl}(\mathbf{x},\mathbf{x_r})\mathbf{z}\\ &=\sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T\Big(\hat{h}_{ijl}(\mathbf{y})\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\\ &+|\tilde{h}_{ijl}(\mathbf{y})-\hat{h}_{ijl}(\mathbf{y})|\mathbf{Y}_{ijl}(\mathbf{x},\mathbf{x_r})\Big)\mathbf{z}, \mathbf{x}\in\Phi_l,\\ &l=1,\,2,\ldots,L. \end{aligned} \tag{46}$$

In every sub-domain, we have the upper and lower linear functions $\overline{h}_{ijl}(\mathbf{y})$ and $\underline{h}_{ijl}(\mathbf{y})$. Through the upper and lower linear functions we can define $\hat{h}_{ijl}(\mathbf{y}) = \frac{1}{2}(\overline{h}_{ijl}(\mathbf{y})+\underline{h}_{ijl}(\mathbf{y}))$, $\hat{h}_{ijl}(\mathbf{y})$ is the estimation of $\tilde{h}_{ijl}(\mathbf{y})$. Then it is always valid that $|\tilde{h}_{ijl}(\mathbf{y})-\hat{h}_{ijl}(\mathbf{y})| \leq \frac{1}{2}(\overline{h}_{ijl}(\mathbf{y})-\underline{h}_{ijl}(\mathbf{y}))$. Then the stability condition (46) can be bounded as follows:

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{h}_{ijl}(\mathbf{y})\mathbf{z}^T\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z} &\leq \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T(\hat{h}_{ijl}(\mathbf{y})\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})+\\ &\frac{1}{2}(\overline{h}_{ijl}(\mathbf{y})-\underline{h}_{ijl}(\mathbf{y}))\mathbf{Y}_{ijl}(\mathbf{x},\mathbf{x_r}))\mathbf{z}\\ &=\sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T(\hat{h}_{ijl}(\mathbf{y})\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})+\delta_{ijl}(\mathbf{y})\mathbf{Y}_{ijl}(\mathbf{x},\mathbf{x_r}))\mathbf{z}<0 \end{aligned} \tag{47}$$

where $\delta_{ijl}(\mathbf{y}) = \frac{1}{2}(\overline{h}_{ijl}(\mathbf{y})-\underline{h}_{ijl}(\mathbf{y}))$.

In order to further relax the stability analysis results, we bring the state information from each sub-domain into the stability analysis. Defining the slack matrices $\mathbf{S}_l(\mathbf{y}) = \mathbf{S}_l^T(\mathbf{y}) \in \Re^{N\times N} \geq 0$, $l = 1, 2,\ldots, L$, it follows from (47) that

$$\begin{aligned}\dot{V}(t) &\leq \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T(\hat{h}_{ijl}(\mathbf{y})\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})+\delta_{ijl}(\mathbf{y})\mathbf{Y}_{ijl}(\mathbf{x},\mathbf{x_r}))\mathbf{z}\\ &\leq \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T\Big(\hat{h}_{ijl}(\mathbf{y})\mathbf{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})+\delta_{ijl}(\mathbf{y})\mathbf{Y}_{ijl}(\mathbf{x},\mathbf{x_r})+(\mathbf{y}-\underline{\mathbf{y}}_l)^T\mathbf{D}(\overline{\mathbf{y}}_l-\mathbf{y})\mathbf{S}_l(\mathbf{y})\Big)\mathbf{z}\end{aligned} \tag{48}$$

where $\underline{\mathbf{y}}_l \in \Re^q$ and $\overline{\mathbf{y}}_l \in \Re^q$ are the lower and upper bound of $\mathbf{y}$ in the l-th sub-domain, $l = 1, 2, \ldots, L$; $\mathbf{D} = \text{diag}\{d_1, d_2, \ldots, d_q\} \in \Re^{q\times q}$ is a diagonal matrix whose element is either 0 or 1. When $d_r = 0$, $r = 1, 2, \ldots, m$, the state information of y_r is not included. Combining the stability condition in (41) with the information of membership functions, the results can be summarized as in the following theorem.

Theorem 2. *The IT2 PFMB system (25), which is formed by a nonlinear plant represented by the IT2 polynomial fuzzy model and the IT2 polynomial fuzzy controller connected in a closed loop, in which the states are driven to follow those of the stable reference model (12) subject to H_∞ performance (43) if there exist polynomial matrices $\mathbf{S}_l(\mathbf{y}) = \mathbf{S}_l(\mathbf{y})^T \in \Re^{N\times N} \geq 0$, $\mathbf{F}_j(\mathbf{h}) \in \Re^{m\times q}$, $\mathbf{G}_j(\mathbf{h}) \in \Re^{m\times q}$, $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$, $\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x_r}) = \mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x_r})^T \in \Re^{N\times N}$, i = 1, 2, …, p, j = 1, 2, …, c, l = 1, 2, …, L, such that the following SOS-based conditions are satisfied:*

$$
\begin{aligned}
&\nu^T(\mathbf{S}_l(\mathbf{y}) - \varepsilon_1(\mathbf{y})\mathbf{I})\nu \text{ is SOS} \quad \forall l;\\
&\nu^T(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_2(\tilde{\mathbf{x}})\mathbf{I})\nu \text{ is SOS};\\
&\rho^T(\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x_r}) - \varepsilon_3(\mathbf{x}, \mathbf{x_r})\mathbf{I})\rho \text{ is SOS} \quad \forall i, j, l;\\
&\rho^T(\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x_r}) - \mathbf{\Xi}_{ij}(\mathbf{x}, \mathbf{x_r}) - \varepsilon_4(\mathbf{x}, \mathbf{x_r})\mathbf{I})\rho \text{ is SOS} \quad \forall i, j, l;\\
-&\rho^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}(\hat{h}_{ijl}(\mathbf{y})\mathbf{\Xi}_{ij}(\mathbf{x}, \mathbf{x_r}) + \delta_{ijl}(\mathbf{y})\mathbf{Y}_{ijl}(\mathbf{x}, \mathbf{x_r}))+\\
&(\mathbf{y} - \underline{\mathbf{y}}_l)^T\mathbf{D}(\overline{\mathbf{y}}_l - \mathbf{y})\mathbf{S}_l(\mathbf{y}) + \varepsilon_5(\mathbf{x}, \mathbf{x_r}, \mathbf{y})\mathbf{I}\Big)\rho \text{ is SOS} \quad \forall l
\end{aligned}
\tag{49}
$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$, $\mathbf{x_y}$ and $\mathbf{y}$, $\rho \in \Re^{2N+m}$ is an arbitrary vector independent of $\mathbf{x}$, $\mathbf{x_r}$ and $\mathbf{y}$, $\hat{h}_{ijl}(\mathbf{y})$ and $\delta_{ijl}(\mathbf{y})$ are linear functions defined by $\overline{h}_{ijl}(\mathbf{y})$ and $\underline{h}_{ijl}(\mathbf{y})$; $\mathbf{D}$ = diag$\{d_1, d_2, \ldots, d_q\} \in \Re^{q\times q}$ is a predefined diagonal matrix; $\varepsilon_1(\mathbf{y}) > 0$, $\varepsilon_2(\tilde{\mathbf{x}}) > 0$, $\varepsilon_3(\mathbf{x}, \mathbf{x_r}) > 0$, $\varepsilon_4(\mathbf{x}, \mathbf{x_r}) > 0$, $\varepsilon_5(\mathbf{x}, \mathbf{x_r}, \mathbf{y}) > 0$ are predefined scalar polynomials for numerical reason; $\underline{\mathbf{y}}_l$ and $\overline{\mathbf{y}}_l$ are the predefined lower and upper bounds of $\mathbf{y}$ in the l-th sub-domain.

Remark 2. *Referring to Theorem 2, the number of SOS variables is $pcL + L + 2c + 1$ and the number of SOS based stability conditions is $2pcL + 2L + 1$.*

It can be seen that although dividing more sub-domains can introduce richer information into the analysis, but the computational burden represented by the number of SOS variables and conditions also increased. Also the parameters $\varepsilon_1(\mathbf{y})$, $\varepsilon_2(\tilde{\mathbf{x}})$, $\varepsilon_3(\mathbf{x}, \mathbf{x_r})$, $\varepsilon_4(\mathbf{x}, \mathbf{x_r})$ and $\varepsilon_5(\mathbf{x}, \mathbf{x_r}, \mathbf{y})$ are small positive values used to just keep the stability conditions strictly positive definite when using the SOS-TOOL toolbox for numerical reasons. δ_{ijl}, $\underline{h}_{ijl}$ and $\overline{h}_{ijl}$ are dependent on the shape of membership functions and the way we extract the information from the membership functions.

4. Simulation Examples

Example 1: To demonstrate the effectiveness of the proposed approach, we design a polynomial fuzzy control system equipped with different model and control fuzzy rules to track the states of the reference using only the system output.

Let us consider a three-rule polynomial fuzzy model with $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \quad x_2]^T$,

$$\mathbf{A}_1(x_1) = \begin{bmatrix} 0.59 - 0.12x_1 & -7.29 - 1.82x_1 \\ 0.01 & -2.85 \end{bmatrix},$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.02 + 2.25x_1 & -4.64 + 0.72x_1 \\ 0.35 & -8.56 \end{bmatrix},$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} 0.73 + 0.45x_1 & 8.45 + 2.13x_1 \\ 0.26 & -15.43 \end{bmatrix},$$

$$\mathbf{B}_1(x_1) = \begin{bmatrix} 1 + 1.35x_1 + 2.33x_1^2 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_2(x_1) = \begin{bmatrix} 8 - 0.62x_1 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_3(x_1) = \begin{bmatrix} 4 - 0.73x_1 + 3.35x_1^2 \\ 0.8 \end{bmatrix},$$

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$$\mathbf{C} = [1 \quad 0].$$

Give that $\mathbf{C} = [1 \quad 0]$, we have the output $\mathbf{y} = \mathbf{C}\hat{\mathbf{x}}(\mathbf{x}) = x_1$ in this simulation. The membership functions are chosen as $\underline{w}_1(x_1) = 1 - 1/(1 + e^{(-x_1+3.5)})$, $\underline{w}_3(x_1) = 1 - 1/(1 + e^{(-x_1-3.5)})$, $\overline{w}_2(x_1) = 1 - \underline{w}_1(x_1) - \underline{w}_3(x_1)$; $\overline{w}_1(x_1) = 1 - 1/(1+e^{(-x_1+2.5)})$, $\overline{w}_3(x_1) = 1-1/(1+e^{(-x_1-2.5)})$, $\underline{w}_2(x_1) = 1-\overline{w}_1(x_1)-\overline{w}_3(x_1)$;

$$\underline{m}_1(x_1) = \begin{cases} 1 & \text{for } x_1 < -5.2 \\ \frac{-x_1+4.8}{10} & \text{for } -5.2 \le x_1 \le 4.8 \\ 0 & \text{for } x_1 > 4.8 \end{cases}, \quad (50)$$

$$\overline{m}_1(x_1) = \begin{cases} 1 & \text{for } x_1 < -4.8 \\ \frac{-x_1+5.2}{10} & \text{for } -4.8 \le x_1 \le 5.2 \\ 0 & \text{for } x_1 > 5.2 \end{cases}, \quad (51)$$

$\underline{m}_2(x_1) = 1 - \overline{m}_1(x_1)$ and $\overline{m}_2(x_1) = 1 - \underline{m}_1(x_1)$.

For the reference model, the system and input matrices are

$$\mathbf{A}_r = \begin{bmatrix} -1.5 & -1 \\ -0.3 & -8.5 \end{bmatrix},$$

$$\mathbf{B}_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the output matrix is $\mathbf{C} = [1 \quad 0]$.

It should be noted in this example that the number of fuzzy rules and the membership functions employed for the polynomial model and the polynomial fuzzy controller are different, which can reduce the controller implementation cost when a less number of membership functions is employed in the controller.

Referring to Theorem 2, we choose $\varepsilon_1(\mathbf{y}) = \varepsilon_2(\tilde{\mathbf{x}}) = \varepsilon_3(\mathbf{x}, \mathbf{x_r}) = \varepsilon_4(\mathbf{x}, \mathbf{x_r}) = \varepsilon_5(\mathbf{x}, \mathbf{x_y}, \mathbf{y}) = 0.001$; $\mathbf{X}(\tilde{\mathbf{x}})$ as a polynomial of degree 0; $\mathbf{M}_j(x_1)$ and $\mathbf{N}_j(x_1)$, $j = 1, 2, \ldots, c$ are polynomials with monomials in x_1 of degree 0, $\mathbf{S}_l(x_1)$ is of degree 0. Throughout this example, the membership functions $\tilde{w}_i(x_1)$ and $\tilde{m}_j(x_1)$ used in the simulations are gained from type reduction in (6) and (19) where $\underline{\lambda}_1(x_1) = (\sin(5x_1) + 1)/2, \overline{\lambda}_1(x_1) = 1 - \underline{\lambda}_1(x_1)$, $\underline{\lambda}_3(x_1) = (\cos(5x_1) + 1)/2, \overline{\lambda}_3(x_1) = 1 - \underline{\lambda}_3(x_1)$, $\tilde{w}_2(x_1) = 1 - \tilde{w}_1(x_1) - \tilde{w}_3(x_1)$, $\underline{\kappa}_j(x_1) = \overline{\kappa}_j(x_1) = 0.5$, $j = 1, 2$. The number of sub-domains used in the simulation is 20, i.e., $L = 20$.

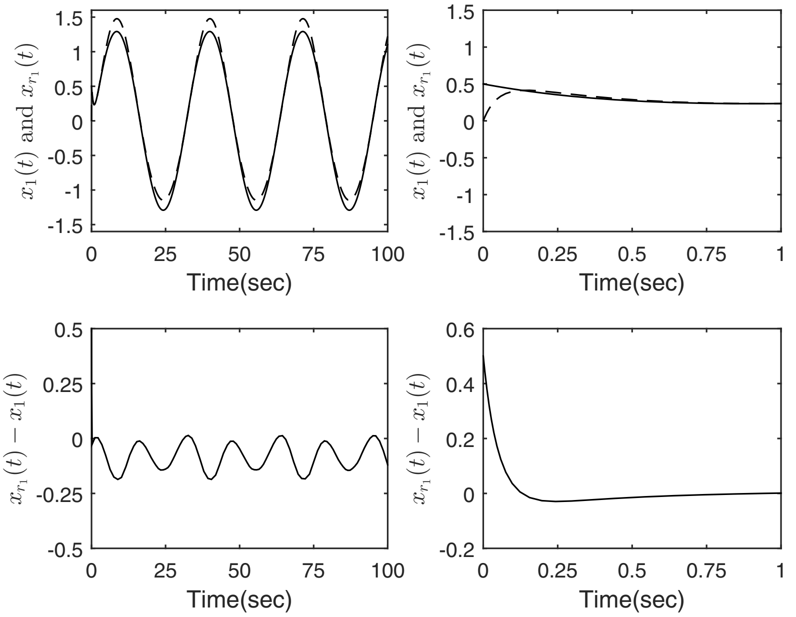

Figure 1: Tracking control performance for $x_1(t)$ with $\sigma_1 = 1.5291$ and $\sigma_2 = 0.3928$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system $(x_1(t))$, and the solid curves are the trajectory of response in the reference model $(x_{r_1}(t))$. The low two sub-figures show the difference between $x_1(t)$ and $x_{r_1}(t)$.

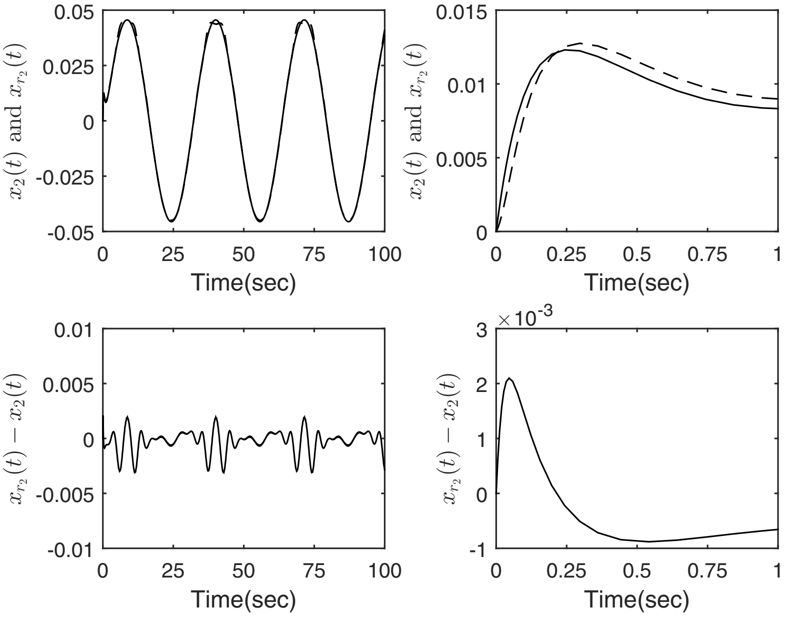

Figure 2: Tracking control performance for $x_2(t)$ with $\sigma_1 = 1.5291$ and $\sigma_2 = 0.3928$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system ($x_2(t)$), and the solid curves are the trajectory of response in the reference model ($x_{r_2}(t)$). The below two sub-figures show the difference between $x_2(t)$ and $x_{r_2}(t)$.

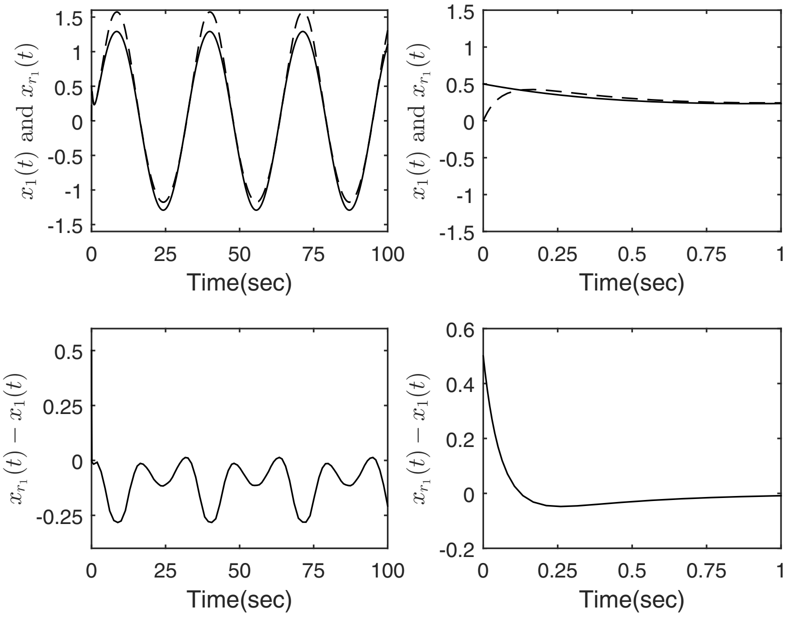

Figure 3: Tracking control performance for $x_1(t)$ with $\sigma_1 = 10$ and $\sigma_2 = 10$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system ($x_1(t)$), and the solid curves are the trajectory of response in the reference model ($x_{r_1}(t)$). The below two sub-figures show the difference between $x_1(t)$ and $x_{r_1}(t)$.

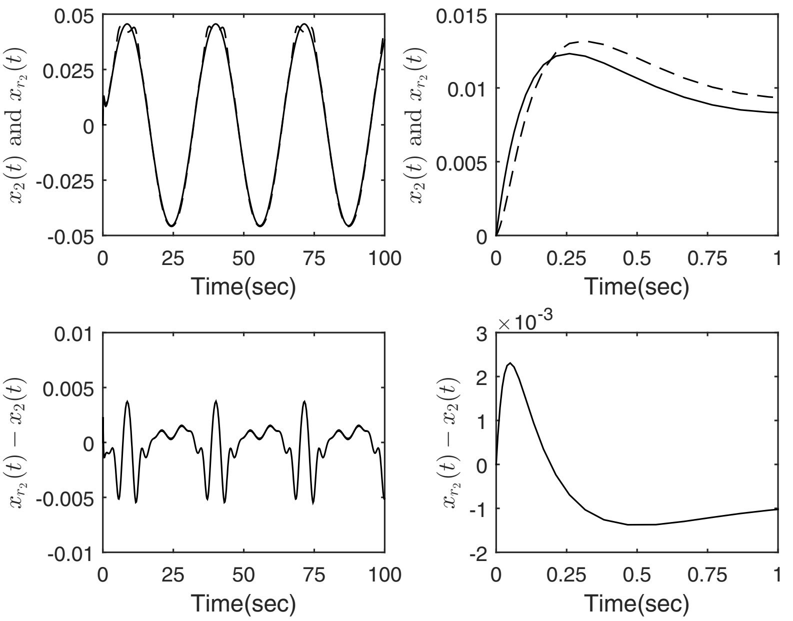

Figure 4: Tracking control performance for $x_2(t)$ with $\sigma_1 = 10$ and $\sigma_2 = 10$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system ($x_2(t)$), and the solid curves are the trajectory of response in the reference model ($x_{r_2}(t)$). The below two sub-figures show the difference between $x_2(t)$ and $x_{r_2}(t)$.

The simulations have been done under two cases according to different sets of H_∞ performance parameters of σ_1 and σ_2. The proposed fuzzy controller is employed to control the nonlinear plant subject to the initial conditions $\mathbf{x}(0) = [0 \quad 0]^T$ and $\mathbf{x_r}(0) = [0.5 \quad 0]^T$.

In the first case, we investigate the tracking performance with the smallest values of σ_1 and σ_2. To obtain the smallest values of σ_1 and σ_2, the summation of σ_1 and σ_2 can be set as the objective function in SOSTOOLS and further be minimized. In the second case, we consider $\sigma_1 = \sigma_2 = 10$ as sufficiently large set of values to investigate the tracking performance for comparison purposes.

Through the two cases, the influence of H_∞ performance parameters σ_1

and σ_2 can be demonstrated through performing time response simulation. For the first case, by solving the solution to Theorem 2, we obtained $\mathbf{X} = \begin{bmatrix} 2.6213 & 0.1441 \\ 0.1441 & 0.2699 \end{bmatrix}$ and the feedback gains as $\mathbf{F}_1 = -3.1016$, $\mathbf{F}_2 = -2.5222$, $\mathbf{G}_1 = 0.0593$, $\mathbf{G}_2 = -0.0575$. The time response simulations are shown in Figs. 1 and 2 under $\sigma_1 = 1.5291$ and $\sigma_2 = 0.3928$.

For the second case, by solving the solution to Theorem 2, we obtained $\mathbf{X} = \begin{bmatrix} 2.0430 & -0.8658 \\ -0.8658 & 3.967 \end{bmatrix}$ and the feedback gains as $\mathbf{F}_1 = -9.2750$, $\mathbf{F}_2 = -6.9686$, $\mathbf{G}_1 = -0.0464$ and $\mathbf{G}_2 = -0.0797$. The time response simulations are shown in Fig. 3 and 4 under $\sigma_1 = 10$ and $\sigma_2 = 10$.

It can be seen from Figs. 1 to 4 that when σ_1 and σ_2 are small, the performance of the tracking control is decent that the states of fuzzy model can track closely the those of the reference model. But when the value of σ_1 and σ_2 are increased to 10 in the second case, the tracking error becomes obvious, especially for $x_1(t)$ in Fig. 3 that the tracking error of $x_1(t)$ is more than 0.25, which is significantly larger than its counterparts in the first cases. In Fig. 4, there are also some high spikes in the error of $x_2(t)$ that some of them are close to 0.005, which are larger than their counterparts in the first case. From the simulation results, it reveals that good tracking performance can be achieved by using smaller values of σ_1 and σ_2, which verify the analysis result.

Remark 3. *When Theorem 1 in [15] is applied to facilitate the stability analysis, there is no feasible solution can be found. It can be seen that through incorporating the information of membership functions into the stability analysis, the analysis results can be further relaxed by adopting Theorem 2.*

Example 2: In this example, the tracking control design of an inverted pendulum will be investigated to verify the effectiveness of the proposed approach. The inverted pendulum is an open-loop unstable nonlinear system, which requires a well-designed controller to stabilize the system and further drive the states of the fuzzy model to track those of the reference model. At first, The dynamic equation of the inverted pendulum [16] is given by

$$\ddot{\theta}(t) = \frac{g\sin(\theta(t)) - am_p S\dot{\theta}(t)^2\sin(2\theta(t))/2 - a\cos(\theta(t))u(t)}{4S/3 - am_p S\cos^2(\theta(t))}, \tag{52}$$

where $\theta(t)$ is the angular displacement of the inverted pendulum, $g = 9.8$ m/s^2, $m_p \in [m_{p_{\min}} \quad m_{p_{\max}}] = [2 \quad 3]$ kg is the mass of the pendulum, $M_c \in [M_{c_{\min}} \quad M_{c_{\max}}] = [8 \quad 16]$ kg is the mass of the cart, $a = \frac{1}{m_p + M_c}$, $2S = 1$m is the length of the pendulum, and $u(t)$ is the force applied on the cart. In the investigation, m_p and M_c are treated as the parameter uncertainties. To transform the dynamic equation of the inverted pendulum into state variable models, $\theta(t)$ and $\dot{\theta}(t)$ are treated as the state variables. Also by considering the uncertainties in the plant, we can construct an IT2 PFMB fuzzy model.

The 4-rule polynomial fuzzy model can be adopted to describe the inverted pendulum as follows:

$$\begin{aligned} &\text{Rule } i : \text{IF } f_1(\mathbf{x}(t)) \text{ is } \tilde{M}_1^i \text{ AND } f_2(\mathbf{x}(t)) \text{ is } \tilde{M}_2^i \\ &\qquad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t), \quad i = 1, 2, 3, 4. \end{aligned} \tag{53}$$

Blending all the fuzzy rules together, we have:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} \tilde{w}_i \Big(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \Big), \tag{54}$$

where

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) = [x_1(t) \quad x_2(t)]^T = [\theta(t) \quad \dot{\theta}(t)]^T,$$

$$x_1(t) \in [\frac{-5\pi}{12} \quad \frac{5\pi}{12}], \quad x_2(t) \in [-5 \quad 5],$$

$$\mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ f_{1_{\min}} & 0 \end{bmatrix}, \quad \mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ f_{1_{\max}} & 0 \end{bmatrix},$$

$$\mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ f_{2_{\min}} \end{bmatrix}, \quad \mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ f_{2_{\max}} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The IT2 membership functions for the fuzzy model are defined as shown in Table 1.

In Table 1, we have

$$
\begin{aligned}
f_1(\mathbf{x}(t)) &= \frac{g - am_pSx_2(t)^2\cos(x_1(t))}{4S/3 - am_pS\cos^2(x_1(t))}\Big(\frac{\sin(x_1(t))}{x_1(t)}\Big),\\
f_2(\mathbf{x}(t)) &= \frac{-a\cos(x_1(t))}{4S/3 - am_pS\cos^2(x_1(t))}.
\end{aligned}
$$

Through the Taylor series based approach proposed in [36], the minimum and maximum values of f_1 and f_2 can be obtained in polynomial functions as follows:

$$
\begin{aligned}
f_{1_{\min}} &= -1.8932x_1^2 + 12.0513, f_{1_{\max}} = -4.3666x_1^2 + 18.4800,\\
f_{2_{\min}} &= -0.0388x_1^4 + 0.1194x_1^2 - 0.1765,\\
f_{2_{\max}} &= -0.0097x_1^4 + 0.0568x_1^2 - 0.0895.
\end{aligned}
$$

The lower and upper grades of membership are respectively defined as:

$$
\begin{aligned}
w_i^L(\mathbf{x}(t)) &= \underline{\mu}_{\tilde{M}_1^i}(\mathbf{x}(t)) \times \underline{\mu}_{\tilde{M}_2^i}(\mathbf{x}(t)),\\
w_i^U(\mathbf{x}(t)) &= \overline{\mu}_{\tilde{M}_1^i}(\mathbf{x}(t)) \times \overline{\mu}_{\tilde{M}_2^i}(\mathbf{x}(t))
\end{aligned}
$$

for all i.

Based the IT2 PFMB fuzzy model, a two-rule IT2 polynomial fuzzy controller is adopted to drive the states of the inverted pendulum to track those of the reference model.

The following two-rule IT2 polynomial fuzzy controller is adopted to describe the inverted pendulum:

$$
\begin{aligned}
&\text{Rule } j : \text{IF } x_1(t) \text{ is } \tilde{N}^j\\
&\qquad \text{THEN } \mathbf{u}(t) = \mathbf{F}_j\mathbf{e}_{\mathbf{y}(t)} + \mathbf{G}_j\mathbf{y}_\mathbf{r}(t), \quad j = 1, 2. \qquad (55)
\end{aligned}
$$

After combining of all the fuzzy rules, we have

$$
\mathbf{u}(t) = \tilde{m}_1(x_1(t))(\mathbf{F}_1\mathbf{e}_{\mathbf{y}(t)} + \mathbf{G}_1\mathbf{y}_\mathbf{r}(t)) + \tilde{m}_2(x_1(t))(\mathbf{F}_2\mathbf{e}_{\mathbf{y}(t)} + \mathbf{G}_2\mathbf{y}_\mathbf{r}(t)), \quad (56)
$$

where $\tilde{m}_1(x_1(t))$ and $\tilde{m}_2(x_1(t))$ are the IT2 membership functions of the polynomial fuzzy controller.

Table 1: Lower and Upper Membership Functions for the Interval Type-2 Fuzzy Model of the Inverted Pendulum.

Lower and upper membership functions	
$\underline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t)))$ $= \frac{f_{1_{\max}} - f_1(\mathbf{x}(t))}{f_{1_{\max}} - f_{1_{\min}}};$ $\overline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^4}(f_1(\mathbf{x}(t)))$ $= \frac{f_1(\mathbf{x}(t)) - f_{1_{\min}}}{f_{1_{\max}} - f_{1_{\min}}};$ with $x_2(t) = 0, m_p = m_{p_{\max}}$ $= 3$kg and $M_c = M_{c_{\min}} = 8$kg	$\underline{\mu}_{\tilde{M}_1^1}(f_2(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^3}(f_2(\mathbf{x}(t)))$ $= \frac{f_{2_{\max}} - f_2(\mathbf{x}(t))}{f_{2_{\max}} - f_{2_{\min}}};$ $\overline{\mu}_{\tilde{M}_1^2}(f_2(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}(t)))$ $= \frac{f_2(\mathbf{x}(t)) - f_{2_{\min}}}{f_{2_{\max}} - f_{2_{\min}}}$ with $m_p = m_{p_{\max}}$ $= 3$kg and $M_c = M_{c_{\max}} = 16$kg
$\overline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t)))$ $= \frac{f_{1_{\max}} - f_1(\mathbf{x}(t))}{f_{1_{\max}} - f_{1_{\min}}};$ $\underline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}(t)))$ $= \frac{f_2(\mathbf{x}(t)) - f_{2_{\min}}}{f_{2_{\max}} - f_{2_{\min}}};$ with $x_2(t) = x_{2_{\max}}, m_p = m_{p_{\max}}$ $= 3$kg and $M_c = M_{c_{\min}} = 8$kg	$\overline{\mu}_{\tilde{M}_1^1}(f_2(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^3}(f_2(\mathbf{x}(t)))$ $= \frac{f_{2_{\max}} - f_2(\mathbf{x}(t))}{f_{2_{\max}} - f_{2_{\min}}}$ $\underline{\mu}_{\tilde{M}_1^2}(f_2(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}(t)))$ $= \frac{f_2(\mathbf{x}(t)) - f_{2_{\min}}}{f_{2_{\max}} - f_{2_{\min}}};$ with $m_p = m_{p_{\min}} = 2$kg and $M_c = M_{c_{\min}} = 8$kg

The upper and lower bounds of the membership functions of the fuzzy controller are defined as follows:

$$\overline{m}_1(x_1(t)) = \begin{cases} 0 & \text{for } x_1(t) < -\frac{5\pi}{12} \\ \frac{x_1(t)+5\pi/12}{5\pi/12} & \text{for } -\frac{5\pi}{12} \le x_1(t) \le 0 \\ \frac{5\pi/12 - x_1(t)}{5\pi/12} & \text{for } 0 \le x_1(t) \le \frac{5\pi}{12} \\ 0 & \text{for } x_1(t) > \frac{5\pi}{12} \end{cases} \tag{57}$$

$$\underline{m}_1(x_1(t)) = \begin{cases} 0 & \text{for } x_1(t) < -\frac{5\pi}{12} \\ \frac{0.9(x_1(t)+5\pi/12)}{5\pi/12} & \text{for } -\frac{5\pi}{12} \le x_1(t) \le 0 \\ \frac{0.9(5\pi/12 - x_1(t))}{5\pi/12} & \text{for } 0 \le x_1(t) \le \frac{5\pi}{12} \\ 0 & \text{for } x_1(t) > \frac{5\pi}{12} \end{cases} \tag{58}$$

$\overline{m}_2(x_1(t)) = 1 - \underline{m}_1(x_1(t))$, $\underline{m}_2(x_1(t)) = 1 - \overline{m}_1(x_1(t))$, and $\tilde{m}_2(x_1(t)) = 1 -$

$\tilde{m}_1(x_1(t))$. The type reductions for the controller $\underline{\kappa}_j(x_1(t)) = \overline{\kappa}_j(x_1(t)) = 0.5$, $j = 1, 2$.

The reference model has been chosen as $\mathbf{A_r} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$, $\mathbf{B_r} = [0; 1]$ and $r(t) = 5\sin(0.3t)$.

During the simulation, the m_p is set as 2.5kg and M_c is set as 12kg. Based on Theorem. 2, the number of sub-domains is 10 and other parameters are set as the same with those in example 1, $\varepsilon_1(\mathbf{y}) = \varepsilon_2(\tilde{\mathbf{x}}) = \varepsilon_3(\mathbf{x}, \mathbf{x_r}) = \varepsilon_4(\mathbf{x}, \mathbf{x_r}) = \varepsilon_5(\mathbf{x}, \mathbf{x_y}, \mathbf{y}) = 0.001$; $\mathbf{X}(\tilde{\mathbf{x}})$ as a polynomial of degree 0; $\mathbf{M}_j(x_1(t))$ and $\mathbf{N}_j(x_1(t))$, $j = 1, 2, \ldots, c$ are polynomials with monomials in x_1 of degree 0, $\mathbf{S}_l(x_1(t))$ is of degree 0. The fuzzy controller is employed to control the nonlinear plant subject to the initial condition $\mathbf{x}(0) = [5\pi/12 \quad 0]$ and $\mathbf{x_r}(0) = [-5\pi/12 \quad 0.05]$. The feedback gains have been obtained as $\mathbf{F}_1 = [38826.8862 \quad 13253.1761]$, $\mathbf{F}_2 = [135635.0154 \quad 45292.0873]$, $\mathbf{G}_1 = [-444.3226 \quad -139.3668]$, $\mathbf{G}_2 = [778.4013 \quad 216.8987]$, and $\mathbf{X} = \begin{bmatrix} 0.2325 & -0.6911 \\ -0.6911 & 2.1691 \end{bmatrix}$. The minimum values of σ_1 and σ_2 have been achieved as 0.0663 and 0.0671, respectively. Also the tracking performance can be viewed in Fig. 5 and 6, it can bee seen that the fuzzy controller is able to drive the system states to follow the reference model closely.

For comparison purposes, the simulation with $\sigma_1 = \sigma_2 = 1.2247$ has also been conducted under the same other conditions. The feedback gains in this case have been obtained as $\mathbf{F}_1 = [1554.7354 \quad 434.3737]$, $\mathbf{F}_2 = [3442.7169 \quad 970.2425]$, $\mathbf{G}_1 = [78.3891 \quad 16.8479]$, $\mathbf{G}_2 = [129.1094 \quad 32.5293]$, $\mathbf{X} = \begin{bmatrix} 0.2763 & -0.8900 \\ -0.6911 & 3.1753 \end{bmatrix}$. The simulation results can be viewed in Figs. 7 and 8, it can be seen that the system states can also follow those of the reference model. However, from Figs. 5 to 8, it is clear that the error of $x_1(t)$ and $x_2(t)$ in Figs. 7 and 8 is much larger than the error in Figs. 5 and 6. Therefore it is verified again that smaller values of σ_1 and σ_2 are able to provide better tracking performance.

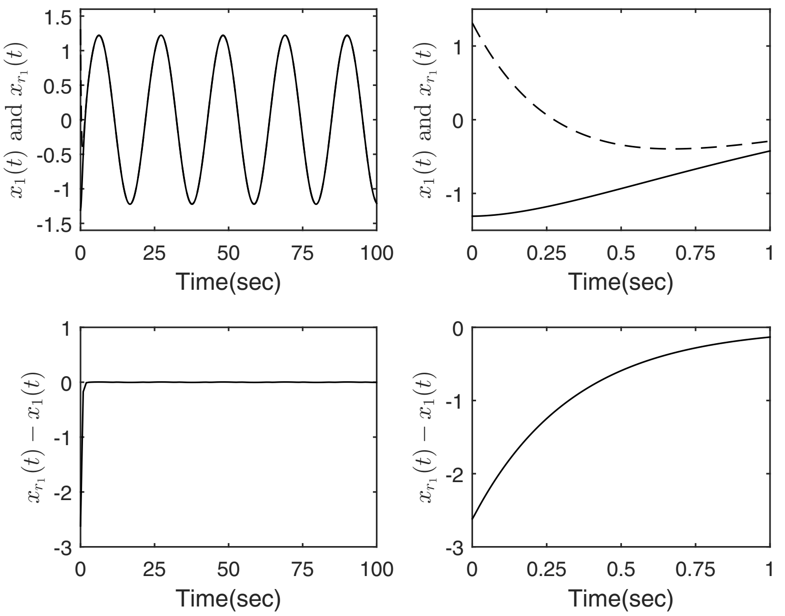

Figure 5: Tracking control performance for $x_1(t)$ with $\sigma_1 = 0.0663$ and $\sigma_2 = 0.0671$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system ($x_1(t)$), and the solid curves are the trajectory of response in the reference model ($x_{r_1}(t)$). The below two sub-figures show the difference between $x_1(t)$ and $x_{r_1}(t)$.

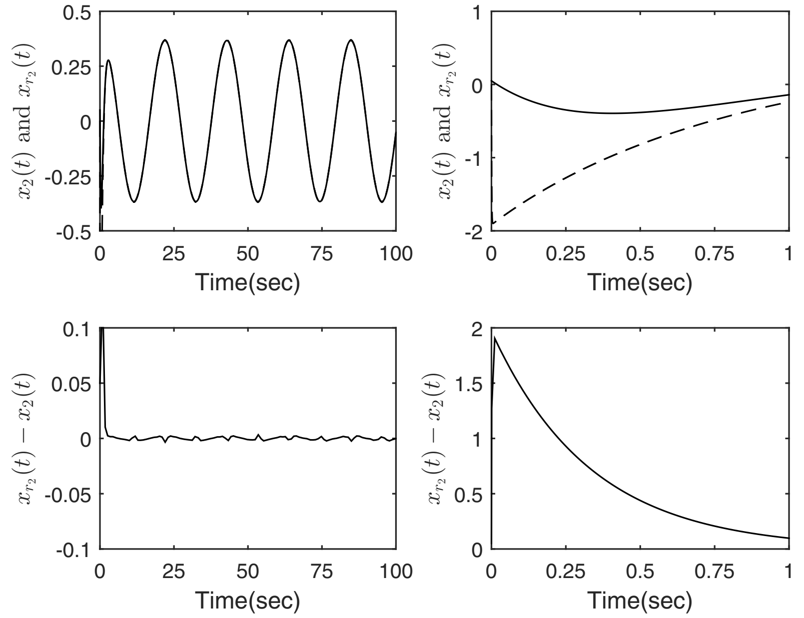

Figure 6: Tracking control performance for $x_2(t)$ with $\sigma_1 = 0.0663$ and $\sigma_2 = 0.0671$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system ($x_2(t)$), and the solid curves are the trajectory of response in the reference model ($x_{r_2}(t)$). The below two sub-figures show the difference between $x_2(t)$ and $x_{r_2}(t)$.

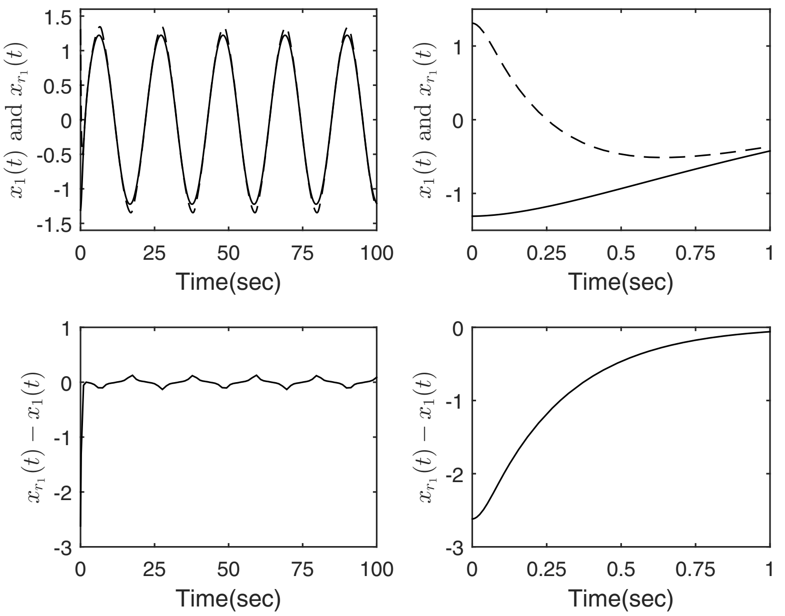

Figure 7: Tracking control performance for $x_1(t)$ with $\sigma_1 = 1.2247$ and $\sigma_2 = 1.2247$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system ($x_1(t)$), and the solid curves are the trajectory of response in the reference model ($x_{r_1}(t)$). The below two sub-figures show the difference between $x_1(t)$ and $x_{r_1}(t)$.

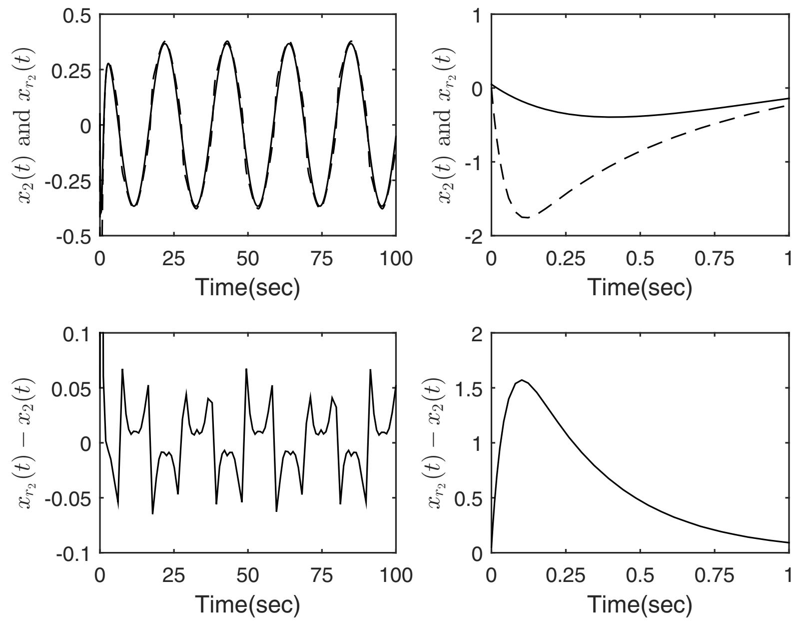

Figure 8: Tracking control performance for $x_2(t)$ with $\sigma_1 = 1.2247$ and $\sigma_2 = 1.2247$. On the top left hand side, the sub-figures shows the simulation time from 0 to 100 seconds, on the top right hand side, the simulation time is from 0 to 1 second. The dashed curves are for the controlled trajectory of the response in the fuzzy system ($x_2(t)$), and the solid curves are the trajectory of response in the reference model ($x_{r_2}(t)$). The below two sub-figures show the difference between $x_2(t)$ and $x_{r_2}(t)$.

Remark 4. *In this example, the PFMB control system equipped with the interval type-2 membership functions have been constructed from a real inverted pendulum case, also the mismatched fuzzy rule sets are adopted in the IT2 polynomial fuzzy model and IT polynomial fuzzy controller, which means the fuzzy model does not share the same fuzzy rules with the fuzzy controller. For the 4-rule IT2 polynomial fuzzy model, we managed to use a 2-rule IT2 polynomial fuzzy controller to drive the states of plant to follow those of the reference model. It can be seen that the tracking performance is good. Also, by using less number of rules and the implementation cost can be reduced, the design flexibility can be enhanced.*

5. Conclusion

In this paper, the tracking control issues based on IT2 PFMB control systems have been investigated. In the analysis, the system output is used for the controller instead of using full states. The mismatched premise membership functions approach has also been adopted to render the control system more flexibly and the information of membership functions is included in the stability analysis to relax the stability conditions. The stability conditions subject to an H_∞ performance are obtained in the form of SOS, which can be solved efficiently through a third-party Matlab toolbox SOSTOOLS. Both numerical simulation example and experimental simulation have been presented to show the effectiveness of the proposed IT2 PFMB control approach. Considering the time-delay and sampled-data issues are frequently encountered in the control applications, time-delay and sampled-data based IT2 PFMB tracking control system will be investigated through the SOS approach in the future, also the newly proposed performance index based on Hankel-norm for fuzzy control systems, which was reported in [39], can be an candidate to improve the tracking performance.

Acknowledgement

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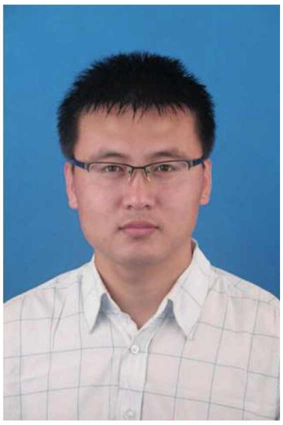
Bo Xiao received bachelor and master (Hons.) degree from Chongqing University, China, in 2010 and 2013, respectively. He is currently working towards his Ph.D. degree at King's College London. His current research interests include fuzzy model based control systems and interval type-2 fuzzy logic and its applications.

Hak-Keung Lam received the B.Eng. (Hons.) and Ph.D. degrees from the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hong Kong, in 1995 and 2000, respectively. During the period of 2000 and 2005, he worked with the Department of Electronic and Information Engineering at The Hong Kong Polytechnic University as Post-Doctoral Fellow and Research Fellow respectively. He joined as a Lecturer at King's College London in 2005 and is currently a Reader.

His current research interests include intelligent control systems and computational intelligence. He has served as a program committee member and international advisory board member for various international conferences and a reviewer for various books, international journals and international conferences. He is an associate editor for *IEEE Transactions on Fuzzy Systems*, *IEEE Transactions on Circuits and Systems II: Express Briefs*, *IET Control Theory and Applications*, *International Journal of Fuzzy Systems* and *Neorocomputing* and editorial board member for a number of journal. He is an IEEE senior member.

He is the coeditor for two edited volumes: *Control of Chaotic Nonlinear Circuits* (World Scientific, 2009) and *Computational Intelligence and Its Applications* (World Scientific, 2012), and the coauthor of the monograph: *Stability Analysis of Fuzzy-Model-Based Control Systems* (Springer, 2011).

Ge Song received her B.Sc degree in automation from Northwestern Polytechnical University, Xi'an, Shaanxi Province, P.R.China, the M.Sc degree in control systems from Imperial College London, London, UK. She is currently working toward the Ph.D. degree in Centre for Robotics Research at King's College London, London, UK. Her research interests are about model-based polynomial fuzzy control systems, especially in interval-type2 fuzzy model based systems.



Hongyi Li received Ph.D. degree in Intelligent Control from the University of Portsmouth, UK, in 2012. He was a Research Associate with the Department of Mechanical Engineering, University of Hong Kong and Hong Kong Polytechnic University. He was a Visiting Principal Fellow with the Faculty of Engineering and Information Sciences, University of Wollongong. He is currently a Professor of the College of Engineering, Bohai University.

Dr. Li received the Best Master Degree Thesis Prize of Liaoning Province in 2010, the Chinese Government Award for Outstanding Student Abroad in 2012 and the Scopus Young Researcher New Star Scientist Award in 2013, the Second Prize of Shandong Natural Science Award in 2014 and the First Prize of Liaoning Natural Science and Technology Academic Achievements Award in 2015. He also won the honor of Liaoning Excellent Talents in University Department of Education Liaoning Province, New Century Excellent Talents in University of Ministry of Education of China and Liaoning Distinguished Professor.

His research interests include fuzzy control, robust control and their applications. He has been in the editorial board of several international journals, including *IEEE Transactions Neural Networks and Learning Systems*, *Neurocomputing* and *Circuits, Systems, and Signal Processing* etc. He has been a Guest Editor of *IET Control Theory and Applications* and *International Journal of Fuzzy Systems*.